Asymptotic behavior of stationary solutions to elastic wave equations in a perturbed half-space (Spectral and Scattering Theory and Related Topics)

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Abstract

We deal with stationary scattering problems for elastic wave equation with a free boundary condition in a local perturbation of the three dimensional half-space. Applying the method of Agmon-Hörmander, we show that all solutions to the stationary elastic wave equations in the Agmon-Hörmander space are characterized in terms of the generalized Fourier transform associated with the elastic operator. Moreover, we investigate asymptotic properties: a uniform asymptotic expansion of the solutions in the averaged sense and representation of the S-matrix. Our expansion describes the behavior of body waves and Rayleigh surface waves in the same topology.

1 Introduction

A basic issue in the scattering theory for partial differential equations is to analyze relations between asymptotic behaviors of solutions at infinity and their Fourier transforms. In particular, the following problems are focused in the stationary scattering theory. Given a self-adjoint operator L associated with the PDE of mathematical physics :

- Construct the Fourier transformation.
- Describe the set of solutions to $(L-\lambda)u=0$ in terms of the Fourier transform.
- Expand the resolvent $R(\lambda \pm i0)$ of L at spatial infinity.
- Expand the solutions to $(L-\lambda)u=0$ at spatial infinity.

The results also play a key role in inverse scattering problems.

An appropriate method to deal with the above problems has already been established by Agmon-Hörmander [1]. Introducing a Besov type function space $\mathcal{B}$ and its conjugate space $\mathcal{B}^*$, they showed that all solutions in $\mathcal{B}^*$ to the equation $(P_0(D)-\lambda)u=0$ are characterized by the Fourier transform restricted to the characteristic surface $\{\xi\,;\,P_0(\xi)=\lambda\}$. This result was extended to two-body

Schrödinger equations (Yafaev [21]), three-body Schrödinger equations (Isozaki [6]) and Laplacians on non-compact manifolds with applications to inverse problems (Isozaki [7] and Isozaki-Kurylev-Lassas [10], see also Isozaki-Kurylev [11]).

We are interested in an elastic wave equation $\mathbf{u}_{tt} + L\mathbf{u} = \mathbf{0}$ with a free boundary condition in $\Omega \subset \mathbb{R}^3$ which is a local perturbation of the half-space $\mathbb{R}^3_+$, where

$$L\mathbf{u} = \left\{ -\frac{1}{\rho(\mathbf{x})} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{u}) \right\}_{1 \le i \le 3}.$$

Here $\mathbf{u} = {}^t(u_1, u_2, u_3)$ is the vector displacement in Ω, $\rho(\mathbf{x}) > 0$ is the density of Ω and the stress tensor $\sigma_{ij}(\mathbf{u})$ has the form

$$\sigma_{ij}(\mathbf{u}) = \lambda(\mathbf{x})(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu(\mathbf{x})\mathcal{E}_{ij}(\mathbf{u}),$$

where $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are Lamé coefficients and $\mathcal{E}_{ij}(\mathbf{u}) = (\partial_i u_j + \partial_j u_i)/2$ is the deformation tensor. This equation has been investigated as a simple model describing the seismic wave propagation (see e.g. [2]).

Time-dependent scattering theory for the elastic wave equation in the half-space have been investigated by Kawashita-Kawasita-Soga ([14], [15]). They constructed translation representations of the Lax-Phillips type for the elastic wave equation in the half-space ([14]) and developed a scattering theory of the Lax-Phillips type for the elastic wave equation in a perturbed half-space. A representation of the scattering kernel was given in [15]. By using the representation, Kawashita-Soga [16] studied a connection between the singular part of the scattering kernel and singularities of the Rayleigh surface wave passing through the perturbed boundary.

With regard to the stationary scattering theory for the elastic operator, although some results—stationary scattering theory in the weighted L^2 space (Dermanjian-Guillot [3]), eigenfunction expansions in stratified media (Shimizu [19]) and existence and uniqueness for the problem of diffraction by an elastic wedge (Kamotski-Lebeau [13])—are known, the characterization of the solutions in $\mathcal{B}^*$ has remained unclear. Moreover, few studies have focused on the asymptotic expansion of the solution uniform with respect to directions. This uniformity is crucial to observe the behavior of the solution near the surface. However, the solution possesses the anisotropy in its spatial asymptotics, which causes a difficulty in deriving the uniform asymptotic expansion.

As is well-known, the solution to the elastic equation is composed of (i) body waves, i.e. P-waves and S-waves propagating inside the body, and (ii) surface waves, i.e. Rayleigeh waves, propagating along the surface (Guillot [5]). Therefore, the generalized eigenfunctions for the elastic operator L in Ω are written as a sum of plane waves: incident P-waves and their reflections, incident S-waves and their reflections and Rayleigh surface waves (see [3]). In their asymptotic expansions, therefore, it is expected that the expansions of P-waves and S-waves (the body waves) involve spherical waves of the form $e^{\pm i\sqrt{\lambda}r}/r$, where $\lambda > 0$ and $r = |\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^3$. On the other hand, the expansions of the Rayleigh surface waves involve spherical waves of the form $e^{-x_3}e^{\pm i\sqrt{\lambda}r_*}/\sqrt{r_*}$, where $r_* = |x_*|$ and $x_* \in \mathbb{R}^2$. Here we used a notation $\mathbb{R}^3 \ni \mathbf{x} = (x_*, x_3), x_* = (x_1, x_2)$.

This is similar to the multi-channel scattering property appearing in the quantum mechanical many-body problems, and in [6], the asymptotic expansion of the resolvent of the 3-body Schr'odinger operator is obtained in $\mathcal{B}$-$\mathcal{B}^*$ spaces. One can compare the 3-cluster scattering to the body wave and the 2-cluster scattering to the surface wave. This suggests us to use the same idea for the elastic equation.

Here, let us mention a difficulty in dealing with the body wave due to the reflection. In the case of the half-space $\mathbb{R}^3_+$, it is known that the reflected body waves have different behavior in each of the following regions:

$$\mathbb{E}^0_{SV} = \left\{ x = (x_*, x_3) \in \mathbb{R}^3_+; 0 < x_3 < \left(\frac{c_P^2}{c_S^2} - 1 \right)^{1/2} |x_*| \right\},$$
$$\mathbb{E}_{SV} = \left\{ x = (x_*, x_3) \in \mathbb{R}^3_+; x_3 > \left(\frac{c_P^2}{c_S^2} - 1 \right)^{1/2} |x_*| \right\}.$$

For example, the reflected SV-wave generated by the incident P-wave does not travel in $\mathbb{E}^0_{SV}$. This phenomenon suggests that the asymptotic expansion of the reflected SV-wave vanishes in $\mathbb{E}^0_{SV}$ and is described by the spherical wave of the form $e^{\pm i\sqrt{\lambda}r}/r$ in $\mathbb{E}_{SV}$.

As such, the generalized eigenfunctions for the elastic operator in $\mathbb{R}^3_+$ have anisotropy in their asymptotic expansions as $|\mathbf{x}| \to \infty$. The difficulty in the uniform asymptotic expansion and the characterization of the solution results from this anisotropy of the solutions.

This paper presents a characterization in the Agmon-Hörmander space $\mathcal{B}^*$ of the solutions to the stationary elastic wave equations in terms of the Fourier transform associated with the elastic operators in $\mathbb{R}^3_+$ and Ω. Our main results are :

- Limiting absorption principle (LAP) on $\mathcal{B}$-$\mathcal{B}^*$ space.
- Uniform asymptotic expansions averaged in $\mathcal{B}^*$ of the resolvent.
- Construction of the generalized Fourier transform for the operator L.
- Characterization of the set of solutions to $(L - \lambda)\mathbf{u} = \mathbf{0}$ in terms of the Fourier transform.
- Asymptotic expansion of the solutions to $(L - \lambda)\mathbf{u} = \mathbf{0}$ at $|\mathbf{x}| \to \infty$ and representation of the S-matrix.

Our expansion of the solutions describes the behavior of body waves and Rayleigh surface wave in the same topology $\mathcal{B}^*$. A technically difficult part of this study is to analyze the resolvent of the elastic operator L because generalized Fourier transform for the operator L has singularities along the cone $\partial\mathbb{E}_{SV}$. We overcome the difficulty by applying the classical stationary phase method (c.f. [17]) and idea used in [6]. This method will be applicable to analyze waves propagating along some surface boundaries (e.g. Stoneley wave).

1.1 Elastic operator in a perturbed half-space

Let $K \subset \mathbb{R}^3$ be a bounded closed set and Ω_- be a bounded open set in $\mathbb{R}^3_-$. We consider an elastic solid $\Omega \subset \mathbb{R}^3$ such that

$$\Omega \cap \{\mathbf{x} \in \mathbb{R}^3 ; |\mathbf{x}| > R\} = \Omega \cap \{\mathbf{x} \in \mathbb{R}^3_+ ; |\mathbf{x}| > R\},$$
$$\Omega = (\mathbb{R}^3_+ \setminus K) \cup \Omega_-,$$

where $R > 0$ is a fixed constant and $\mathbb{R}^3_+ = \{\mathbf{x} \in \mathbb{R}^3; x_3 > 0\}$. We assume that Ω satisfies the cone condition.

Let $W^{k,p}(\Omega)$ be the usual Sobolev space of order k in $L^p(\Omega)$. Suppose that the Lamé coefficients $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$, and the density $\rho(\mathbf{x})$ satisfy the following:

(A1) λ, μ, $\rho \in W^{1,\infty}(\Omega)$.

(A2) There exist positive constants m and M satisfying

$$0 < m \leq \lambda(\mathbf{x}),\ \rho(\mathbf{x}),\ \mu(\mathbf{x}) \leq M$$

for $\mathbf{x} \in \bar{\Omega}$.

(A3) Let λ_0 and μ_0 be the two Lamé parameters in a homogeneous, isotropic, elastic half-space. We denote the density of the elastic half-space by ρ_0. Suppose that

$$\lambda(\mathbf{x}) = \lambda_0, \quad \mu(\mathbf{x}) = \mu_0, \quad \rho(\mathbf{x}) = \rho_0$$

for $\mathbf{x} \in B_R^e = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| > R\}$.

(A4) There exist positive constans $C > 0$ and $a > 0$ such that

$$e^{a|\mathbf{x}|}\,|D_x^\alpha(\lambda(\mathbf{x}) - \lambda_0)| \leq C$$

for $|\alpha| \leq 1$ and $\mathbf{x} \in \Omega$.

We consider an inhomogeneous isotropic elastic medium which occupies the perturbed half-space Ω. As is well known, the stress tensor $\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))_{1\leq i,j\leq 3}$ is given by

$$\sigma_{ij}(\mathbf{u}) = \lambda(\mathbf{x})(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu(\mathbf{x})\mathcal{E}_{ij}(\mathbf{u})$$

and the deformation tensor $\mathcal{E}(\mathbf{u}) = (\mathcal{E}_{ij}(\mathbf{u}))_{1\leq i,j\leq 3}$ is given by

$$\mathcal{E}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where δ_{ij} is the Kronecker's delta and $\mathbf{u} = {}^t(u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$ is the displacement at position $\mathbf{x} \in \Omega$.

Put

$$(\mathcal{L}\mathbf{u})_i = -\frac{1}{\rho(\mathbf{x})}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{u}).$$

Consider the elastic operator

$$\mathcal{L}\mathbf{u} = -\frac{1}{\rho}\operatorname{div}\sigma(\mathbf{u}) = \{(\mathcal{L}\mathbf{u})_i\}_{i=1,2,3}$$

with a boundary condition $\sigma(\mathbf{u})\boldsymbol{\nu}|_{\partial\Omega} = \mathbf{0}$, where $\boldsymbol{\nu}$ is the exterior normal at $\mathbf{x} \in \partial\Omega$. Here the trace $\sigma(\mathbf{u})\boldsymbol{\nu}|_{\partial\Omega} = \mathbf{0}$ means the following generalized sense:

$$\int_\Omega (\mathcal{L}\mathbf{u})_i \bar{\mathbf{v}}_i \rho(\mathbf{x})\,d\mathbf{x} - \int_\Omega \left\{\lambda(\mathbf{x})(\nabla\cdot\mathbf{u})(\nabla\cdot\bar{\mathbf{v}}) + 2\mu(\mathbf{x})\mathcal{E}_{ij}(\mathbf{u})\mathcal{E}_{ij}(\bar{\mathbf{v}})\right\} d\mathbf{x} = 0 \tag{1.1}$$

for $\mathbf{u} \in H^1(\Omega, \mathbb{C}^3) \cap L^2(\Omega, \mathcal{L}, \mathbb{C}^3)$ and $\mathbf{v} \in H^1(\Omega, \mathbb{C}^3)$, where

$$L^2(\Omega, \mathcal{L}, \mathbb{C}^3) = \{\mathbf{u} \in L^2(\Omega, \mathbb{C}^3)\,;\, \mathcal{L}\mathbf{u} \in L^2(\Omega, \mathbb{C}^3)\}$$

and $H^m(\Omega,\mathbb{C}^3)=W^{m,2}(\Omega,\mathbb{C}^3)$.

As in [3], elastic operator $L\mathbf{u}=\mathcal{L}\mathbf{u}$ in $L^2(\Omega,\mathbb{C}^3,\rho(x)dx)$ with a domain

$$D(L)=\{\mathbf{u}\in H^1(\Omega,\mathbb{C}^3)\cap L^2(\Omega,\mathcal{L},\mathbb{C}^3)\,;\,\sigma(\mathbf{u})\boldsymbol{\nu}|_{\partial\Omega}=\mathbf{0}\}$$

is a positive self-adjoint operator, the spectrum $\sigma(L)$ is $[0,\infty)$, continuous spectrum $\sigma_{\mathrm{cont}}(L)=[0,\infty)$, and continuous singular spectrum $\sigma_{\mathrm{sc}}(L)=\emptyset$. In addition, the elastic operator L has no positive eigenvalues embedded in $(0,\infty)$ under the assumptions (A1), (A2), and (A4) (see Sini [20]). Thus the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(L)$ is $[0,\infty)$.

1.2 Main results

Before stating our main theorems, let us introduce some notations. Put $B_R=\{\mathbf{x}\in\mathbb{R}^3,|\mathbf{x}|<R\}$. Let us define the $\mathcal{B}$-$\mathcal{B}^*$ space in Ω as

$$\mathcal{B}=\mathcal{B}(\Omega,\mathbb{C}^3)=\{\ \mathbf{u}\in L^2_{\mathrm{loc}}(\Omega,\mathbb{C}^3)\ ;\ \|\mathbf{u}\|_{\mathcal{B}}<\infty\}$$

with

$$\|\mathbf{u}\|_{\mathcal{B}}=\sum_{j=0}^{\infty}2^{j/2}\|\mathbf{u}\|_{L^2(\Omega_j)},\quad \Omega_j=\{\mathbf{x}\in\mathbb{R}^3\,;\,r_{j-1}<|\mathbf{x}|<r_j\}\cap\Omega,$$

where $r_j=2^j(j\geq 0)$, $r_{-1}=0$. Then the norm of the dual space $\mathcal{B}^*$ is equivalent to

$$\mathbf{u}\in\mathcal{B}^*\Longleftrightarrow\|\mathbf{u}\|_{\mathcal{B}^*}=\sup_{R\geq 1}\left(\frac{1}{R}\int_{\Omega\cap B_R}|\mathbf{u}(\mathbf{x})|^2d\mathbf{x}\right)^{1/2}<\infty.$$

The closure $\mathcal{B}^*_0$ of $L^2(\Omega)$ in the norm of $\mathcal{B}^*$ consists of functions $\mathbf{u}(\mathbf{x})$ satisfying

$$\lim_{R\to\infty}\frac{1}{R}\int_{\Omega\cap B_R}|\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}=0.$$

Note that the relation between the space $\mathcal{B}$ and the weighted L^2-space is as follows:

$$L^{2,s}\subset\mathcal{B}\subset L^{2,1/2}\subset L^2\subset L^{2,-1/2}\subset\mathcal{B}^*_0\subset\mathcal{B}^*\subset L^{2,-s}$$

for $s>1/2$. Here the weighted L^2 space $L^{2,s}$ is defined as

$$\mathbf{u}\in L^{2,s}(\Omega)\Longleftrightarrow\|\mathbf{u}\|_s=\|(1+|\mathbf{x}|)^2\mathbf{u}\|_{L^2(\Omega)}.$$

Let $\mathbb{S}^2_+=\{\mathbf{x}\in\mathbb{R}^3_+,|\mathbf{x}|=1\}$ and $\mathbb{S}^1=\{x_*\in\mathbb{R}^2,|x_*|=1\}$. The following result is for the construction of the generalized Fourier transform.

Theorem 1.1. *For $\lambda\in(0,\infty)$, there exists a bounded operator*

$$\mathcal{F}(\lambda)\,:\,\mathcal{B}\longrightarrow\boldsymbol{h}:=\left[L^2(\mathbb{S}^2_+)\right]^3\times L^2(\mathbb{S}^1)$$

having the following properties:

1. *$\mathcal{F}(\lambda)$ defined by $(\mathcal{F}\mathbf{f})(\lambda) = \mathcal{F}(\lambda)\mathbf{f}$ is uniquely extended to a partial isometric operator with initial set $\mathcal{H}_{\rm ac}(L)$ (the absolutely continuous subspace for L), and final set*

$$\hat{\mathcal{H}} = \begin{pmatrix} L^2((0,\infty)\,;\, L^2(\mathbb{S}^2_+)\,;\, \rho(\lambda_P)d\lambda) \\ L^2((0,\infty)\,;\, L^2(\mathbb{S}^2_+)\,;\, \rho(\lambda_S)d\lambda) \\ L^2((0,\infty)\,;\, L^2(\mathbb{S}^2_+)\,;\, \rho(\lambda_S)d\lambda) \\ L^2((0,\infty)\,;\, L^2(\mathbb{S}^1)\,;\, \rho_1 d\lambda) \end{pmatrix},$$

where $\rho(\lambda_\sharp) = \frac{\sqrt{\lambda}}{2c_\sharp}$, $\sharp = P, S$ and $\rho_1 = \frac{1}{2}$. Moreover $\mathcal{F}$ diagonalizes L:

$$(\mathcal{F}L\mathbf{f})(\lambda) = \lambda(\mathcal{F}\mathbf{f})(\lambda), \qquad {}^\forall\lambda \in (0,\infty), \quad {}^\forall\mathbf{f} \in D(L).$$

2. *For $\mathbf{f} \in \mathcal{H}_{\rm ac}(L)$, the inversion formula holds:*

$$\begin{aligned} \mathbf{f} = {}& \text{s-}\lim_{N\to\infty} \int_{1/N}^{N} \mathcal{F}_P(\lambda)^*(\mathcal{F}_P\mathbf{f})(\lambda)\rho(\lambda_P)\, d\lambda \\ &+ \sum_{\flat = SV, SH} \text{s-}\lim_{N\to\infty} \int_{1/N}^{N} \mathcal{F}_\flat(\lambda)^*(\mathcal{F}_\flat\mathbf{f})(\lambda)\rho(\lambda_S)\, d\lambda \\ &+ \text{s-}\lim_{N\to\infty} \int_{1/N}^{N} \mathcal{F}_R(\lambda)^*(\mathcal{F}_R\mathbf{f})(\lambda)\rho_1\, d\lambda, \end{aligned}$$

where $\mathcal{F}_P(\lambda), \mathcal{F}_{SV}(\lambda), \mathcal{F}_{SH}(\lambda), \mathcal{F}_R(\lambda)$ are the components of $\mathcal{F}(\lambda)$.

3. *Let us denote the set of bounded operators from a set $\mathcal{A}$ to a set $\mathcal{B}$ by $B(\mathcal{A}, \mathcal{B})$. Then for $\lambda \in (0,\infty)$, we have $\mathcal{F}(\lambda) \in B(\mathcal{B}; \boldsymbol{h})$ and $\mathcal{F}(\lambda)^* \in B(\boldsymbol{h}; \mathcal{B})$. Moreover, the operator $\mathcal{F}(\lambda)^*$ is an eigenoperator of L in the sense that $(L-\lambda)\mathcal{F}(\lambda)^*\mathbf{f} = \mathbf{0}$ for any $\mathbf{f} \in \boldsymbol{h}$.*

We next state a result of the asymptotic expansion of the resolvent. In order to state the result, let us define vectors $\mathbf{d}_P(\omega)$, $\mathbf{d}_{SV}(\omega)$ and $\mathbf{d}_{SH}(\omega)$ as

$$\mathbf{d}_P(\omega) = \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in \mathbb{S}^2_+, \quad \mathbf{d}_{SV}(\omega) = \begin{pmatrix} \left(\frac{\omega_3}{|\omega_*|}\right)\omega_1 \\ \left(\frac{\omega_3}{|\omega_*|}\right)\omega_2 \\ |\omega_*| \end{pmatrix}, \quad \mathbf{d}_{SH}(\omega) = \begin{pmatrix} -\frac{\omega_2}{|\omega_*|} \\ \frac{\omega_1}{|\omega_*|} \\ 0 \end{pmatrix}$$

for $\omega \in \mathbb{S}^2_+$ and define $\mathbf{d}_R^{(\ell)}(\nu)$, $\ell = 1, 2$, as

$$\mathbf{d}_R^{(1)}(\nu) = \begin{pmatrix} -i\nu_1 \\ i\nu_2 \\ \tilde{c}_{RP} \end{pmatrix}, \qquad \mathbf{d}_R^{(2)}(\nu) = \begin{pmatrix} -i\nu_1\tilde{c}_{RS} \\ i\nu_2\tilde{c}_{RS} \\ -1 \end{pmatrix}$$

for $\nu \in \mathbb{S}^1$.

For a vector $\mathbf{a} = \{a_i\}_{i=1,2,3}$ in $\mathbb{R}^3$, we put $\hat{\mathbf{a}} = {}^t(a_1, a_2, -a_3)$. Then, vectors $\mathbf{d}_P$, $\mathbf{d}_{SV}$ and $\mathbf{d}_{SH}$ are orthonormal bases in $\mathbb{R}^3$.

It should be mentioned that for a direction of the wave propagation $\omega = (\omega_*, \omega_3) \in \mathbb{S}^2_+$, vectors $\mathbf{d}_P(\omega)$, $\mathbf{d}_{SV}(\omega)$ and $\mathbf{d}_{SH}(\omega)$ are expressed as directions of P-wave displacement, SV-wave displacement, SH-wave displacement, respectively.

Let c_P, c_S, c_R be propagation speeds of P-wave, S-wave and Rayleigh wave in $\mathbb{R}^3_+$. In what follows, the asymptotic relation $\mathbf{u} \simeq \mathbf{v}$ means that

$$\lim_{R\to\infty} \frac{1}{R} \int_{B_R^+} |\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} = 0, \qquad B_R^+ = \{\mathbf{x} \in \mathbb{R}^3_+ \, ; \, |\mathbf{x}| < R\}. \tag{1.2}$$

Theorem 1.2. *Let $\lambda > 0$. Then the following follows:*

1. *For any $\lambda \in (0,\infty)$, the limit*

$$\lim_{\varepsilon\to 0} (R(\lambda \pm i\varepsilon)\mathbf{f}, \mathbf{g}) := (R(\lambda \pm i0)\mathbf{f}, \mathbf{g}), \qquad {}^\forall \mathbf{f}, \mathbf{g} \in \mathcal{B}$$

   exists.

2. *There exists a constant $C > 0$ such that*

$$\|R(\lambda \pm i0)\mathbf{f}\|_{\mathcal{B}^*} \le C\|\mathbf{f}\|_{\mathcal{B}}, \qquad \lambda \in (0,\infty).$$

3. *For $\mathbf{f} \in \mathcal{B}$ and $\lambda \in (0,\infty)$, the boundary value of the resolvent of L admits the following asymptotic expansion*

$$\begin{aligned} R(\lambda + i0)\mathbf{f} \simeq {} & C\frac{e^{i\sqrt{\lambda}r/c_P}}{r}(\mathcal{F}_P(\lambda)\mathbf{f})\mathbf{d}_P(\varphi) + C\frac{e^{i\sqrt{\lambda}r/c_S}}{r}(\mathcal{F}_{SV}(\lambda)\mathbf{f})\hat{\mathbf{d}}_{SV}(\varphi) \\ & + C\frac{e^{i\sqrt{\lambda}r/c_S}}{r}(\mathcal{F}_{SH}(\lambda)\mathbf{f})\mathbf{d}_{SH}(\varphi) \\ & + \sum_{\ell=1}^{2} \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} E_\ell (\mathcal{F}_R(\lambda)\mathbf{f})\mathbf{d}_R^{(\ell)}(\varphi_*), \end{aligned}$$

   where $C = e^{i\pi/4}(2\rho_0)^{-1/2}$ and, τ_ℓ and E_ℓ are constants depending only on c_P, c_S and c_R.

This theorem shows that the resolvent $R(\lambda + i0)$ is expanded in terms of the generalized Fourier transform $\mathcal{F}(\lambda)$, in addition, its expansion describes the behavior of Rayleigh surface waves near the boundary.

We finish this subsection by the characterizing the stationary $\mathcal{B}^*$-solutions to $(L-\lambda)\mathbf{u} = 0$ and giving the asymptotic expansion of the $\mathcal{B}^*$-solutions. In order to state the results, let us introduce a function space $\boldsymbol{h}(\lambda)$:

$$\begin{aligned} \boldsymbol{h}(\lambda) &= \{\varphi \in \boldsymbol{h} \, : \, \|\varphi\|_{\boldsymbol{h}(\lambda)} < \infty\}, \\ \|\varphi\|^2_{\boldsymbol{h}(\lambda)} &= \rho(\lambda_P)\|\varphi_P\|^2_{L^2(\mathbb{S}^2_+)} + \rho(\lambda_S)\|\varphi_{SV}\|^2_{L^2(\mathbb{S}^2_+)} \\ &\quad + \rho(\lambda_S)\|\varphi_{SH}\|^2_{L^2(\mathbb{S}^2_+)} + \rho_1\|\varphi_R\|^2_{L^2(\mathbb{S}^1)}, \end{aligned}$$

where $\rho(\lambda_\flat) = \frac{\sqrt{\lambda}}{2c_\flat}$, $\flat = P, S$ and $\rho_1 = \frac{1}{2}$. We note that

$$\|\mathbf{f}\|_{\hat{\mathcal{H}}} = \int_0^\infty \|\mathbf{f}(\lambda)\|^2_{\boldsymbol{h}(\lambda)} \, d\lambda.$$

Theorem 1.3. *Let $\lambda \in (0,\infty)$. Suppose that* $\mathbf{u}$ *satisfies* $(L-\lambda)\mathbf{u} = \mathbf{0}$. *Then* $\mathbf{u} \in \mathcal{B}^*$ *if and only if* $\mathbf{u} = \mathcal{F}(\lambda)^*\Theta$ *for some* $\Theta \in \boldsymbol{h}(\lambda)$.

Theorem 1.4. *Let $\lambda \in (0,\infty)$. Suppose that* $\mathbf{u} \in \mathcal{B}^*$ *satisfies* $(L-\lambda)\mathbf{u} = \mathbf{0}$. *Then there exists* $\mathbf{f}^{(\pm)} = (f_P^{(\pm)}, f_{SV}^{(\pm)}, f_{SH}^{(\pm)}, f_R^{(\pm)}) \in \mathbf{h}(\lambda)$ *such that*

$$
\begin{aligned}
\mathbf{u}(\mathbf{x}) \simeq\ & \frac{e^{i\sqrt{\lambda}r/c_P}}{r} f_P^{(+)}\mathbf{d}_P(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(+)}\mathbf{d}_{SV}(\varphi) + \frac{e^{i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(+)}\mathbf{d}_{SH}(\varphi) \\
& + \sum_{\ell=1}^{2} \frac{e^{i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} E_\ell f_R^{(+)} \mathbf{d}_R^{(\ell)}(\varphi_*) \\
& - \frac{e^{-i\sqrt{\lambda}r/c_P}}{r} f_P^{(-)}\mathbf{d}_P^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SV}^{(-)}\mathbf{d}_{SV}^{(-)}(\varphi) - \frac{e^{-i\sqrt{\lambda}r/c_S}}{r} f_{SH}^{(-)}\mathbf{d}_{SH}^{(-)}(\varphi) \\
& - \sum_{\ell=1}^{2} \frac{e^{-i\sqrt{\lambda}r_*/c_R}}{\sqrt{r_*}} e^{-\sqrt{\lambda}\tau_\ell x_3} \overline{E_\ell} f_R^{(-)} \mathbf{d}_R^{(\ell)}(-\varphi_*),
\end{aligned}
$$

where $\mathbf{d}_\sharp^{(-)}(\varphi) = \mathbf{d}_\sharp(-\varphi_*, \varphi_3)$ $(\sharp = P, SV, SH)$ *and,* $\tau_\ell > 0$ *and* E_ℓ *are constants depending only on* c_P, c_S *and* c_R. *Moreover,*

$$
r = |\mathbf{x}|, \quad \varphi = \frac{\mathbf{x}}{r}, \quad r_* = |x_*|, \quad \varphi_* = \frac{x_*}{r_*}
$$

for $\mathbb{R}^3_+ \ni \mathbf{x} = (x_*, x_3)$. *Furthermore, there exists a unitary operator* $\boldsymbol{S}(\lambda)$ *on* $\mathbf{h}(\lambda)$ *such that* $\mathbf{f}^{(+)} = \boldsymbol{S}(\lambda)\mathbf{f}^{(-)}$.

The operator $\boldsymbol{S}(\lambda)$ is called the S-matrix. It will be shown that $\boldsymbol{S}(\lambda)$ is unitary equivalent to the S-matrix constructed by means of the time-dependent method. Remember that body waves are three-dimensional spherical waves propagating in the elastic body Ω; Rayleigh wave is a two-dimensional spherical wave propagating along the boundary and it exponentially decays in a direction to x_3. This is a reason that the asymptotic analysis of the solutions to the elastic wave equation is difficult. Our theorem shows that the leading term of the asymptotic expansion of the $\mathcal{B}^*$-solutions is described as a sum of spherical P-waves, S-waves, and Rayleigh waves in the same topology (c.f. [6]).

1.3 Structure of this paper

In section 2, we summarize our results on the asymptotics of solutions to stationary elastic wave equation in $\mathbb{R}^3_+$. After introducing generalized eigenfunctions for elastic operator L_0 in $\mathbb{R}^3_+$, the generalized Fourier transform for L_0 are constructed. We show that this Fourier transform is a bounded operator from $\mathcal{B}$ to $\boldsymbol{h}$. In addition, applying the stationary phase method on the sphere, we derive an asymptotic expansion of the Fourier transform in $\mathcal{B}^*$.

Next by using the Mourre theory, the limiting absorption principle in the $\mathcal{B}$-$\mathcal{B}^*$ space is proved. In order to obtain asymptotic expansions for the resolvent $R_0(\lambda \pm i0)$, we apply the uniform stationary phase method (e.g. Lewis [17]) to integral representations for $R_0(\lambda \pm i0)$. Because the generalized Fourier transforms have singularities along the cone $\partial\mathbb{S}_{SV}$, the usual stationary phase method is not applicable.

From these results and the stationary scattering theory, we establish the characterization of $\mathcal{B}^*$-solutions to $(L_0 - \lambda)\mathbf{u} = \mathbf{0}$ and we give an uniform asymptotic expansion of the $\mathcal{B}^*$- solutions.

Asymptotic behavior in a neighborhood of the boundary $\partial\mathbb{E}^0_{SV}$ is described in terms of a Fresnel type integral. As was done by Lewis [17], this method is valid to obtain a leading term of the uniform expansion of an integral with a stationary point near the endpoint of the integral. However, in our case, the remainder terms of the expansion become infinite at the boundary $\partial\mathbb{E}^0_{SV}$. The introduction of $\mathcal{B}^*$-space makes it possible to remove this divergence difficulty.

2 Elastic waves in $\mathbb{R}^3_+$

In this section, we study asymptotic properties of the solutions to stationary elastic wave equation in a homogeneous, isotropic, elastic half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3\,;\, x_3 > 0\}$ with a free boundary. Let ρ_0 be the density of elastic half-space $\mathbb{R}^3_+$. We denote the two Lamé constants by λ_0 and μ_0. Note that body-wave speeds c_P and c_S in elastic half-space $\mathbb{R}^3_+$ are represented as

$$c_P^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \qquad c_S^2 = \frac{\mu_0}{\rho_0}.$$

We consider the elastic operator

$$L_0\mathbf{u} = -\frac{\lambda_0 + \mu_0}{\rho_0}\nabla(\nabla\cdot\mathbf{u}) - \frac{\mu_0}{\rho_0}\Delta\mathbf{u}$$

in $\mathbb{R}^3_+$ with a domain

$$D(L_0) = \left\{\mathbf{u} \in H^1(\mathbb{R}^3_+, \mathbb{C}^3)\,;\, L_0\mathbf{u} \in \mathcal{H}_0,\ \sigma^0(\mathbf{u})\boldsymbol{\nu}_0\big|_{\partial\mathbb{R}^3_+} = \mathbf{0}\right\},$$

where the stress tensor $\sigma^0(\mathbf{u})$ of the elastic half-space $\mathbb{R}^3_+$ is a 3×3 matrix with the (ij)th entry

$$\sigma^0_{ij}(\mathbf{u}) \doteq \lambda_0(\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu_0\mathcal{E}_{ij}(\mathbf{u})$$

and $\boldsymbol{\nu}_0$ is the exterior normal at $\mathbf{x} \in \partial\mathbb{R}^3_+$. Here we used a notation $\mathcal{H}_0 = L^2(\mathbb{R}^3_+, \mathbb{C}^3, \rho_0 d\mathbf{x})$. Note that the trace $\sigma^0(\mathbf{u})\boldsymbol{\nu}_0\big|_{\partial\mathbb{R}^3_+} = \mathbf{0}$ means generalized sense (1.1). As in [3], the elastic operator L_0 is an absolutely continuous positive self-adjoint operator in $\mathcal{H}_0$ whose spectrum is $[0,\infty)$.

2.1 Generalized eigenfunctions in $\mathbb{R}^3_+$

Although generalized eigenfunctions for L_0 were given in [3], we use representation of those given in [12] due to convenience for our study. In order to describe the generalized eigenfunctions, We introduce some notations. Let $\{j,\ell\} = \{P,S,R\}$. The ratio of speed c_j to speed c_ℓ is denoted by $c_{j\ell} = c_j/c_\ell$. For $\mathbf{k} \in \mathbb{R}^3_+$, we put $\mathbf{k} = |\mathbf{k}|\omega, \omega \in \mathbb{S}^2_+$ and

$$\xi_{j\ell}(\mathbf{k}) = \left(c_{j\ell}^2|\mathbf{k}|^2 - |k_*|^2\right)^{1/2}, \gamma_{j\ell}(\omega_*) = \left(c_{j\ell}^2 - |\omega_*|^2\right)^{1/2},$$

$$\xi'_{j\ell}(\mathbf{k}) = \left(|k_*|^2 - c_{j\ell}|\mathbf{k}|^2\right)^{1/2}, \gamma'_{j\ell}(\omega_*) = \left(|\omega_*|^2 - c_{j\ell}\right)^{1/2},$$

where $\omega_* = \frac{k_*}{|\mathbf{k}|}$. Transformations $\zeta_{j\ell}$ and $\tilde{\zeta}_{SP}$ are defined as

$$\zeta_{j\ell} : \omega = \begin{pmatrix}\omega_*\\ \omega_3\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_*\\ \gamma_{j\ell}(\omega_*)\end{pmatrix},$$

$$\tilde{\zeta}_{SP} : \omega = \begin{pmatrix}\omega_*\\ \omega_3\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_*\\ i\gamma'_{SP}(\omega_*)\end{pmatrix}.$$

We note that

$$\mathbf{d}_P(\zeta_{SP}(\omega)) = \begin{pmatrix} \omega_* \\ \gamma_{SP}(\omega_*) \end{pmatrix}, \quad \hat{\mathbf{d}}_{SV}(\zeta_{PS}(\omega)) = \begin{pmatrix} \frac{\gamma_{PS}(\omega_*)}{|\omega_*|}\omega_* \\ -|\omega_*| \end{pmatrix}.$$

Letting $C_\rho = (2\pi)^{-3/2}\rho_0^{-1/2}$, generalized eigenfunctions $\mathbf{\Phi}_P$, $\mathbf{\Phi}_{SV}$, $\mathbf{\Phi}_{SV}^0$, $\mathbf{\Phi}_{SH}$ and $\mathbf{\Phi}_R$ are given as follows:

1. For $(\mathbf{x}, \mathbf{k}) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$ and $\omega = \mathbf{k}/|\mathbf{k}|$,

$$\begin{aligned}\mathbf{\Phi}_P(\mathbf{x}, \mathbf{k}) = C_\rho e^{ik_*\cdot x_*}\Big\{&e^{-ik_3x_3}\hat{\mathbf{d}}_P(\omega) + e^{i\xi_{PS}(\mathbf{k})x_3}\eta_P^{(2)}(\omega)\hat{\mathbf{d}}_{SV}(\zeta_{PS}(\omega)) \\ &-e^{ik_3x_3}\eta_P^{(3)}(\omega)\mathbf{d}_P(\omega)\Big\},\end{aligned}$$

   where

$$\eta_P^{(2)}(\omega) = \frac{4|\omega_*|(c_{PS}^2 - 2|\omega_*|^2)\omega_3}{(c_{PS}^2 - 2|\omega_*|^2)^2 + 4|\omega_*|^2\omega_3\gamma_{PS}(\omega_*)},$$
$$\eta_P^{(3)}(\omega) = \frac{(c_{PS}^2 - 2|\omega_*|^2)^2 - 4|\omega_*|^2\omega_3\gamma_{PS}(\omega_*)}{(c_{PS}^2 - 2|\omega_*|^2)^2 + 4|\omega_*|^2\omega_3\gamma_{PS}(\omega_*)}.$$

2. For $(\mathbf{x}, \mathbf{k}) \in \mathbb{R}^3_+ \times \mathbb{E}_{SV}$ and $\omega = \mathbf{k}/|\mathbf{k}$,

$$\begin{aligned}\mathbf{\Phi}_{SV}(\mathbf{x}, \mathbf{k}) = C_\rho e^{ik_*\cdot x_*}\Big\{&e^{-ik_3x_3}\mathbf{d}_{SV}(\omega) + e^{ik_3x_3}\eta_{SV}^{(2)}(\omega)\hat{\mathbf{d}}_{SV}(\omega) \\ &+e^{i\xi_{SP}(\mathbf{k})x_3}\eta_{SV}^{(3)}(\omega)\mathbf{d}_P(\zeta_{SP}(\omega))\Big\},\end{aligned}$$

   where

$$\eta_{SV}^{(2)}(\omega) = \frac{(1 - 2|\omega_*|^2)^2 - 4|\omega_*|^2\omega_3\gamma_{SP}(\omega_*)}{(1 - 2|\omega_*|^2)^2 + 4|\omega_*|^2\omega_3\gamma_{SP}(\omega_*)},$$
$$\eta_{SV}^{(3)}(\omega) = \frac{4|\omega_*|(1 - 2|\omega_*|^2)\omega_3}{(1 - 2|\omega_*|^2)^2 + 4|\omega_*|^2\omega_3\gamma_{SP}(\omega_*)}.$$

3. For $(\mathbf{x}, \mathbf{k}) \in \mathbb{R}^3_+ \times \mathbb{E}^0_{SV}$ and $\omega = \mathbf{k}/|\mathbf{k}|$,

$$\begin{aligned}\mathbf{\Phi}^0_{SV}(\mathbf{x}, \mathbf{k}) = C_\rho e^{ik_*\cdot x_*}\Big\{&e^{-ik_3x_3}\mathbf{d}_{SV}(\omega) + e^{ik_3x_3}\eta_{SV}^{(5)}(\omega)\hat{\mathbf{d}}_{SV}(\omega) \\ &+e^{-\xi'_{SP}(\mathbf{k})x_3}\eta_{SV}^{(6)}(\omega)\mathbf{d}_P(\tilde{\zeta}_{SP}(\omega))\Big\},\end{aligned}$$

   where

$$\eta_{SV}^{(5)}(\omega) = \frac{(1 - 2|\omega_*|^2)^2 - 4i|\omega_*|^2\omega_3\gamma'_{SP}(\omega_*)}{(1 - 2|\omega_*|^2)^2 + 4i|\omega_*|^2\omega_3\gamma'_{SP}(\omega_*)},$$
$$\eta_{SV}^{(6)}(\omega) = \frac{4|\omega_*|(1 - 2|\omega_*|^2)\omega_3}{(1 - 2|\omega_*|^2)^2 + 4i|\omega_*|^2\omega_3\gamma'_{SP}(\omega_*)}.$$

4. For $(\mathbf{x}, \mathbf{k}) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$ and $\omega = \mathbf{k}/|\mathbf{k}|$,

$$\mathbf{\Phi}_{SH}(\mathbf{x}, \mathbf{k}) = C_\rho e^{ik_*\cdot x_*}(e^{ik_3x_3} + e^{-ik_3x_3})\mathbf{d}_{SH}(\omega).$$

5. Let c_R be the Rayleigh wave speed. Note that $c_R < c_S < c_P$. For $(\mathbf{x}, p) \in \mathbb{R}^3_+ \times \mathbb{R}^2$ and $\nu = p/|p|$,

$$\mathbf{\Phi}_R(\mathbf{x}, p) = \frac{\mathcal{N}_R}{2\pi}|p|^{\frac{1}{2}} e^{ip\cdot x_*} \left\{ R_1 e^{-|p|\tilde{c}_{RP} x_3} \mathbf{d}_R^{(1)}(\nu) + R_2 e^{-|p|\tilde{c}_{RS} x_3} \mathbf{d}_R^{(2)}(\nu) \right\},$$

where

$$R_1 = 2 - c_{RS}^2, \qquad R_2 = -2\tilde{c}_{RP}$$

and $\mathcal{N}_R$ is a normalizeing constant such that

$$4\pi^2 \int_0^\infty |\Psi_R(\mathbf{x}, p)|^2 \rho_0 dx_3 = 1.$$

Here we used notations $\tilde{c}_{R\sharp} = \sqrt{1 - c_{R\sharp}^2}$ $(\sharp = P, S)$.

A function $\mathbf{\Phi}_P$ describes reflection of plane P-wave at a free surface $\partial\mathbb{R}^3_+$. More precisely, the function $\mathbf{\Phi}_P$ consists of a sum of

$$\text{incident P-wave: } e^{ik_*\cdot x_*} e^{-ik_3 x_3} \hat{\mathbf{d}}_P(\omega), \qquad \text{reflected P-wave: } -e^{ik\cdot x} \eta_P^{(3)}(\omega) \mathbf{d}_P(\omega),$$
$$\text{reflected S-wave: } e^{ik_*\cdot x_*} e^{i\xi_{PS}(\mathbf{k}) x_3} \eta_P^{(2)}(\omega) \hat{\mathbf{d}}_{SV}(\zeta_{PS}(\omega)).$$

Similarly to $\mathbf{\Phi}_P$, functions $\mathbf{\Phi}_{SV}$ and $\mathbf{\Phi}_{SH}$ describe reflection of S-wave, in addition, $\mathbf{\Phi}_R$ represents Rayleigh surface wave. Here we should mention that the function $e^{ik_*\cdot x_*} e^{-\xi'_{SP}(\mathbf{k}) x_3} \eta_{SV}^{(6)}(\omega) \mathbf{d}_P(\tilde{\zeta}_{SP}(\omega))$ given in the representation of $\mathbf{\Phi}^0_{SV}$ describes reflected P-wave propagating in a region $\mathbb{E}^0_{SV}$ and exponentially decays in a direction to x_3. In other words, the reflected P-wave generated by incident SV-wave in the region $\mathbb{E}^0_{SV}$ travels along the boundary; this wave is termed evanescent wave.

2.2 Generalized Fourier transform for L_0

We denote the characteristic function of a set A as $\chi(A)$. As is shown in [3], letting

$$(U_0^P \mathbf{u})(\mathbf{k}) = \int_{\mathbb{R}^3_+} \overline{\mathbf{\Phi}_P(\mathbf{x}, \mathbf{k})} \cdot \mathbf{u}(\mathbf{x}) \rho_0 d\mathbf{x},$$
$$(U_0^{SV} \mathbf{u})(\mathbf{k}) = \int_{\mathbb{R}^3_+} \left\{ \chi(\mathbb{E}_{SV}) \overline{\mathbf{\Phi}_{SV}(\mathbf{x}, \mathbf{k})} + \chi(\mathbb{E}^0_{SV}) \overline{\mathbf{\Phi}^0_{SV}(\mathbf{x}, \mathbf{k})} \right\} \cdot \mathbf{u}(\mathbf{x}) \rho_0 d\mathbf{x},$$
$$(U_0^{SH} \mathbf{u})(\mathbf{k}) = \int_{\mathbb{R}^3_+} \overline{\mathbf{\Phi}_{SH}(\mathbf{x}, \mathbf{k})} \cdot \mathbf{u}(\mathbf{x}) \rho_0 d\mathbf{x},$$
$$(U_0^R \mathbf{u})(p) = \int_{\mathbb{R}^3_+} \overline{\mathbf{\Phi}_R(\mathbf{x}, p)} \cdot \mathbf{u}(\mathbf{x}) \rho_0 d\mathbf{x}$$

for $\mathbf{u} \in C_0^\infty(\mathbb{R}^3_+, \mathbb{C}^3)$, then operators $U_0^\flat$ $(\flat = P, SV, SH)$ are extended as a partial isometric operator from $\mathcal{H}_0$ to $L^2(\mathbb{R}^3_+)$. Similarly, U_0^R is extended as an partially isometric operator from $\mathcal{H}_0$ to $L^2(\mathbb{R}^2)$. Thus the Fourier transform U_0 associated with the elastic operator L_0 can be defined as

$$U_0 \mathbf{u} = {}^t(U_0^P \mathbf{u}, U_0^{SV} \mathbf{u}, U_0^{SH} \mathbf{u}, U_0^R \mathbf{u}), \qquad \mathbf{u} \in \mathcal{H}_0.$$

The Fourier transform U_0 is a unitary operator

$$U_0 : \mathcal{H}_0 \longrightarrow \hat{\mathcal{H}}_0 = \left[L^2(\mathbb{R}^3_+;\mathbb{C}^3)\right]^3 \times L^2(\mathbb{R}^2;\mathbb{C}^3)$$

such that

$$U_0(L_0\mathbf{u}) = \begin{pmatrix} c_P^2|\mathbf{k}|^2U_0^P\mathbf{u} \\ c_S^2|\mathbf{k}|^2U_0^{SV}\mathbf{u} \\ c_S^2|\mathbf{k}|^2U_0^{SH}\mathbf{u} \\ c_R^2|p|^2U_0^R\mathbf{u} \end{pmatrix}$$

for $\mathbf{u} \in D(L_0)$ (see Theorem 3.6 in [3]).

We now consider the restriction of the Fourier transform onto the upper half-sphere. Let us define operators $U_0^\sharp$ ($\sharp = P, SV, SH, R$) as

$$(U_0^P(\lambda)\mathbf{f})(\omega) = (U_0^P\mathbf{f})(\sqrt{\lambda_P}\omega), \quad (U_0^\flat(\lambda)\mathbf{f})(\omega) = (U_0^\flat\mathbf{f})(\sqrt{\lambda_S}\omega) \quad \omega \in \mathbb{S}^2_+, \quad \flat = SV, SH,$$
$$(U_0^R(\lambda)\mathbf{f})(\nu) = (U_0^R\mathbf{f})(\sqrt{\lambda_R}\nu), \quad \nu \in \mathbb{S}^1,$$

where $\lambda > 0$ and $\sqrt{\lambda_\sharp} = \sqrt{\lambda}/c_\sharp$. These operators $U_0^\sharp(\lambda)$ are well defined for $\mathbf{f} \in C_0^\infty(\mathbb{R}^3_+, \mathbb{C}^3)$. The operator $U_0^\sharp(\lambda)^*$ ($\sharp = P, SV, SH, R$) formally adjoint to $U_0^\sharp(\lambda)$ is given by the formula

$$\left(U_0^P(\lambda)^*f\right)(\mathbf{x}) = \int_{\mathbb{S}^2_+} \mathbf{\Phi}_P(\mathbf{x}, \sqrt{\lambda_P}\omega)f(\omega)\,d\omega,$$
$$\left(U_0^R(\lambda)^*g\right)(\mathbf{x}) = \int_{\mathbb{S}^1} \mathbf{\Phi}_R(\mathbf{x}, \sqrt{\lambda_R}\nu)g(\nu)\,d\nu$$

for any $f \in L^2(\mathbb{S}^2_+)$ and $g \in L^2(\mathbb{S}^1)$, respectively. In the same way, operators $U_0^{SV}(\lambda)^*$ and $U_0^{SH}(\lambda)^*$ are given.

The following statement shows that the vector-valued operator $U_0(\lambda)$ and $U_0(\lambda)^*$ can be extended to bounded operators from $\mathcal{B}$ to $\boldsymbol{h}$ and from $\boldsymbol{h}$ to $\mathcal{B}^*$; we denote them as $B(\mathcal{B}; \boldsymbol{h})$ and $B(\boldsymbol{h}; \mathcal{B}^*)$, respectively.

Theorem 2.1. *For any $\lambda > 0$, we have*

$$U_0(\lambda) \in B(\mathcal{B}; \boldsymbol{h}), \qquad U_0(\lambda)^* \in B(\boldsymbol{h}; \mathcal{B}^*),$$

where

$$U_0(\lambda)\mathbf{f} = \begin{pmatrix} U_0^P(\lambda)\mathbf{f} \\ U_0^{SV}(\lambda)\mathbf{f} \\ U_0^{SH}(\lambda)\mathbf{f} \\ U_0^R(\lambda)\mathbf{f} \end{pmatrix},$$

for $\mathbf{f} \in \mathcal{B}$ and

$$\begin{aligned} U_0(\lambda)^*\mathbf{g} &= \left(U_0^P(\lambda)^*, U_0^{SV}(\lambda)^*, U_0^{SH}(\lambda)^*, U_0^R(\lambda)^*\right) \begin{pmatrix} g_P \\ g_{SV} \\ g_{SH} \\ g_R \end{pmatrix} \\ &= \sum_{\sharp = P, SV, SH, R} U_0^\sharp(\lambda)^* g_\sharp \end{aligned}$$

for $\mathbf{g} \in \boldsymbol{h}$. *Moreover, following estimates hold for* $\lambda > 0$:

$$\|U_0^\sharp(\lambda)\mathbf{f}\|_{L^2(\mathbb{S}^2_+)} \leqq C\lambda^{-1/2}\|\mathbf{f}\|_{\mathcal{B}}, \tag{2.1}$$

$$\|U_0^R(\lambda)\mathbf{f}\|_{L^2(\mathbb{S}^1)} \leqq C\lambda^{-1/4}\|\mathbf{f}\|_{\mathcal{B}}, \tag{2.2}$$

and

$$\|U_0^\sharp(\lambda)^* f\|_{\mathcal{B}^*} \leqq C\lambda^{-1/2}\|f\|_{L^2(\mathbb{S}^2_+)},$$
$$\|U_0^R(\lambda)^* g\|_{\mathcal{B}^*} \leqq C\lambda^{-1/4}\|g\|_{L^2(\mathbb{S}^1)},$$

where the positive constant C *does not depend on* λ.

A standard argument allows us to prove

Theorem 2.2. *For* $\boldsymbol{f}$, $\boldsymbol{g} \in \mathcal{B}$,

$$(\boldsymbol{f}, \boldsymbol{g}) = \rho(\lambda_\flat)\left(U_0(\lambda)\boldsymbol{f}, U_0(\lambda)\boldsymbol{g}\right)_{\boldsymbol{h}},$$

where $\rho(\lambda_\sharp) = \frac{\sqrt{\lambda}}{2c_\sharp}$, $\sharp = P, S$ *and* $\rho_1 = \frac{1}{2}$.

We next give an asymptotic expansion for $U_0(\lambda)^*\mathbf{g}$. Let us introduce some notations. We denote a reflection operator J as

$$J : \begin{cases} f(\varphi) \longrightarrow f(-\varphi_*, \varphi_3) & \text{for } \varphi \in \mathbb{S}^2_+, \\ g(\nu) \longrightarrow g(-\nu) & \text{for } \nu \in \mathbb{S}^1. \end{cases}$$

We define the scaling $K_{j\ell}$, $\{k, \ell\} = \{P, S\}$ as

$$(K_{j\ell}g)(\varphi) = g(\varphi_{j\ell}) = g(c_{j\ell}\varphi_*, \sqrt{1 - |c_{j\ell}\varphi_*|^2})$$

for $\varphi \in \mathbb{S}^2_+$. Remember that constants $\mathcal{N}_R$, R_1, R_2 and $\tilde{c}_{R\sharp}$, $\sharp = P, S$ are introduced in section 2.1. We set

$$\mathbb{S}_{SV} = \left\{ \omega = (\omega_*, \omega_3) \in \mathbb{S}^2_+ \ ; \ \omega_3 > \left(\frac{c_P^2}{c_S^2} - 1\right)^{1/2} |\omega_*| \right\},$$

$$\mathbb{S}^0_{SV} = \left\{ \omega = (\omega_*, \omega_3) \in \mathbb{S}^2_+ \ ; \ 0 < \omega_3 < \left(\frac{c_P^2}{c_S^2} - 1\right)^{1/2} |\omega_*| \right\}.$$

Theorem 2.3. *For $g = {}^t(g_P, g_{SV}, g_{SH}, g_R) \in h$ and for $\lambda > 0$,*

$$\begin{aligned}
U_0(\lambda)^* g \simeq\ & \frac{C_{\rho_0}}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_P} r}}{r} \left\{ (\eta_P^{(3)} g_P)(\varphi) + c_{SP}^2 \eta_P^{(2)}(\varphi) g_{SV}(\varphi_{SP}) \right\} \mathbf{d}_P(\varphi) \\
& + \frac{C_{\rho_0}}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S} r}}{r} \Big\{ \chi(\mathbb{S}_{SV}) c_{PS}^2 \eta_{SV}^{(3)}(\varphi) f_P(\varphi_{PS}) \\
& + \left(\chi(\mathbb{S}_{SV}) \eta_{SV}^{(2)}(\varphi) + \chi(\mathbb{S}_{SV}^0) \eta_{SV}^{(5)}(\varphi) \right) f_{SV}(\varphi) \Big\} \hat{\mathbf{d}}_{SV}(\varphi) \\
& + \frac{C_{\rho_0}}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S} r}}{r} \left(g_{SH} \mathbf{d}_{SH} \right)(\varphi) \\
& + D \sum_{\ell=1}^{2} \frac{e^{i\sqrt{\lambda_R} r_*}}{\sqrt{r_*}} e^{-\sqrt{\lambda_R} \tau_\ell x_3} R_\ell \left(g_R \mathbf{d}_R^{(\ell)} \right)(\varphi_*) \\
& + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_P}} \frac{e^{-i\sqrt{\lambda_P} r}}{r} J \left(g_P \hat{\mathbf{d}}_P \right)(\varphi) + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S} r}}{r} J \left(g_{SV} \mathbf{d}_{SV} \right)(\varphi) \\
& + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S} r}}{r} J \left(g_{SH} \mathbf{d}_{SH} \right)(\varphi) + \overline{D} \sum_{\ell=1}^{2} \frac{e^{-i\sqrt{\lambda_R} r_*}}{\sqrt{r_*}} e^{-\sqrt{\lambda_R} \tau_\ell x_3} R_\ell J \left(g_R \mathbf{d}_R^{(\ell)} \right)(\varphi_*),
\end{aligned}$$

where

$$C_{\rho_0} = \frac{e^{-i\pi/2}}{\sqrt{2\pi\rho_0}}, \quad D = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \mathcal{N}_R, \qquad \tau_1 = \tilde{c}_{RP}, \quad \tau_2 = \tilde{c}_{RS}.$$

The proof is based on the following stationary phase method for L^2-functions on the sphere (see e.g. [7]).

Lemma 2.1. *Let $\mu > 0$. Then for any $\varphi \in L^2(\mathbb{S}^{n-1})$,*

$$\int_{\mathbb{S}^{n-1}} e^{\pm i\mu x \cdot \omega} \varphi(\omega)\, d\omega \simeq C \frac{e^{i\mu r}}{(\mu r)^{(n-1)/2}} \varphi(\pm \hat{x}) + \overline{C} \frac{e^{-i\mu r}}{(\mu r)^{(n-1)/2}} \varphi(\mp \hat{x}), \tag{2.3}$$

where $r = |x|$, $\hat{x} = x/r$ and $C = e^{-(n-1)\pi i/4}(2\pi)^{(n-1)/2}$.

Lemma 2.2. *Assume that $a(x, \omega) \in C^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ satisfies*

$$|\partial_x^\alpha \partial_\omega^\beta a(x, \omega)| \le C_{\alpha\beta} \langle x \rangle^{-|\alpha|}.$$

Then for any $\varphi \in L^2(\mathbb{S}^{n-1})$,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} e^{i\mu x \cdot \omega} a(x, \omega) \varphi(\omega)\, d\omega \simeq\ & C \frac{e^{i\mu r}}{(\mu r)^{(n-1)/2}} a(x, \hat{x}) \varphi(\hat{x}) \\
& + \overline{C} \frac{e^{-i\mu r}}{(\mu r)^{(n-1)/2}} a(x, -\hat{x}) \varphi(-\hat{x}),
\end{aligned} \tag{2.4}$$

where $r = |x|$, $\hat{x} = x/r$ and $C = e^{-(n-1)\pi i/4}(2\pi)^{(n-1)/2}$.

2.3 Uniform boundedness for the resolvent and asymptotic expansions

In [3], the limiting absorption principle (LAP) for the resolvent of the elastic operator L_0 was proved in a weighted L^2 space, by passing to the partial Fourier transform, and reducing the issue to a boundary value problem for a 1-dimensional operator. In this subsection, we prove LAP by adapting Mourre's commutator calculus to the boundary value problem.

Theorem 2.4. *Let $R_0(z) = (L_0 - z)^{-1}$. Then we have*

1. *For any $\lambda \in (0,\infty)$, the limit*

$$\lim_{\varepsilon\to 0}(R_0(\lambda \pm i\varepsilon)\mathbf{f},\mathbf{g}) := (R_0(\lambda \pm i0)\mathbf{f},\mathbf{g}), \qquad {}^\forall \mathbf{f},\mathbf{g} \in \mathcal{B}$$

   exists and $R_0(\lambda \pm i0) \in B(\mathcal{B};\mathcal{B}^*)$.

2. *There exists a constant $C > 0$ such that*

$$\|R_0(\lambda \pm i0)\mathbf{f}\|_{\mathcal{B}^*} \leq C\|\mathbf{f}\|_{\mathcal{B}}, \qquad \lambda \in (0,\infty).$$

We next show that the resolvent $R_0(\lambda \pm i0)$ is expanded in terms of the Generalized Fourier transform $U_0(\lambda)$. Each terms of asymptotics correspond respectively to the outgoing body waves (P-wave and S-wave) and Rayleigh surface waves.

Theorem 2.5. *Let $\lambda > 0$. Then for any $\mathbf{f} \in \mathcal{B}$ we have*

$$\begin{aligned}(R_0(\lambda + i0)\mathbf{f})(\mathbf{x}) \simeq\ & C(P)\frac{e^{i\sqrt{\lambda_P}r}}{r}((U_0^P(\lambda)\mathbf{f})\mathbf{d}_P)(\varphi)\\ &+ C(S)\frac{e^{i\sqrt{\lambda_S}r}}{r}((U_0^{SV}(\lambda)\mathbf{f})\hat{\mathbf{d}}_{SV})(\varphi)\\ &+ C(S)\frac{e^{i\sqrt{\lambda_S}r}}{r}((U_0^{SH}(\lambda)\mathbf{f})\mathbf{d}_{SH})(\varphi)\\ &+ C(R)\sum_{\ell=1}^{2}\frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}e^{-\sqrt{\lambda}\tau_\ell x_3}R_\ell((U_0^R(\lambda)\mathbf{f})\mathbf{d}_R^{(\ell)})(\varphi_*),\end{aligned}$$

where $\varphi = \mathbf{x}/r$, $r = |\mathbf{x}|$, $\varphi_* = x_*/r_*$, $r_* = |x_*|$ and

$$C(\flat) = \frac{1}{c_\flat^2}\sqrt{\frac{\pi}{2\rho_0}},\ \flat = P, S, \qquad C(R) = \frac{1}{c_R^2}\sqrt{\frac{\pi}{2}}\mathcal{N}_R e^{i\pi/4}.$$

Remark 1. Taking complex conjugate, one can easily obtain the asymptotic expansion of $R_0(\lambda - i0)\mathbf{f}$.

The strategy of the proof is as follows:

- We split the resolvent $R_0(z)\mathbf{f}$ into two parts:

$$R_0(z)\mathbf{f} = \Phi_1(L_0)R_0(z)\mathbf{f} + \Phi_2(L_0)R_0(z)\mathbf{f}, \qquad z = \lambda + i\varepsilon,\ \varepsilon > 0,$$

  where $\Phi_1 \in C_0^\infty(\mathbb{R})$ is a cut-off function with support in a neighborhood of $\mathrm{Re}z = \lambda$; a function Φ_2 is defined as $\Phi_2 = 1 - \Phi_1$.

- We show that
$$\Phi_2(L_0)R_0(z)\mathbf{f} \in \mathcal{B}_0^*$$
for any $\mathbf{f} \in C_0^\infty(\mathbb{R}^3_+;\mathbb{C}^3)$. The proof of this estimate will be found in [9]

- In order to evaluate the leading term of the asymptotic expansion of $R_0(\lambda + i0)\mathbf{f}$, we rewrite the resolvent $R_0(z)\mathbf{f}$ into the following integral form:
$$R_0(z)\boldsymbol{f} = \sum_{\sharp=P,SV,SH,R} B_\sharp(z)\boldsymbol{f},$$
where
$$\begin{aligned}
(B_P(z)\mathbf{f})(\mathbf{x}) &= \left\{(U_0^P)^*\left(\frac{(U_0^P\mathbf{f})(\mathbf{k})}{(c_P^2|\mathbf{k}|^2 - z)}\right)\right\}(\mathbf{x}) \\
&= C_{\rho_0}\sum_{l=1}^{3}\int_0^\infty \frac{\mu^2}{c_P^2\mu^2 - z}(J_{P,l}(\mu)\hat{f}_P)(\mathbf{x})\,d\mu \\
(B_{SV}(z)\mathbf{f})(\mathbf{x}) &= \left\{(U_0^{SV})^*\left(\frac{(U_0^{SV}\mathbf{f})(\mathbf{k})}{(c_S^2|\mathbf{k}|^2 - z)}\right)\right\}(\mathbf{x}) \\
&= C_{\rho_0}\sum_{l=1}^{6}\int_0^\infty \frac{\mu^2}{c_S^2\mu^2 - z}(J_{SV,l}(\mu)\hat{f}_{SV})(\mathbf{x})\,d\mu \\
(B_{SH}(z)\mathbf{f})(\mathbf{x}) &= \left\{(U_0^{SH})^*\left(\frac{(U_0^{SH}\mathbf{f})(\mathbf{k})}{(c_S^2|\mathbf{k}|^2 - z)}\right)\right\}(\mathbf{x}) \\
&= C_{\rho_0}\sum_{l=l}^{3}\int_0^\infty \frac{\mu^2}{c_S^2\mu^2 - z}(J_{SH,l}(\mu)\hat{f}_{SH})(\mathbf{x})\,d\mu \\
(B_R(z)\mathbf{f})(\mathbf{x}) &= \left\{(U_0^R)^*\left(\frac{(U_0^R\mathbf{f})(p)}{(c_R^2|p|^2 - z)}\right)\right\}(\mathbf{x}) \\
&= \sum_{\ell=1}^{2} R_\ell\int_0^\infty \frac{\mu\sqrt{\mu}}{c_R^2\mu^2 - z}e^{-\mu\tau_\ell x_3}(J_{R,\ell}(\mu)\hat{f}_R)(\mathbf{x})\,d\mu,
\end{aligned}$$
and $J_{\sharp,l}(z)$ are integral operator over the partial sphere given in subsection 2.2. Here we abbreviated $U_0^\sharp\mathbf{f}$ as $\hat{f}_\sharp$.

In order to give asymptotic expansions of $B_\sharp(z)\mathbf{f}$, we first evaluate the asymptotic expansion of $J_{\sharp,\ell}(z)$. Next we apply the residue theorem to the integral and take limit as $\varepsilon \to 0$ by using the uniform estimate on the resolvent. In order to evaluate the asymptotic expansion of $J_{\sharp,\ell}(z)$, we use a uniform stationary phase method in [17] instead of the usual stationary phase method because we need to obtain the asymptotic expansion where the stationary point lie in a neighborhood of the boundary of the integral regions.

2.4 Asymptotics of solutions in $\mathbb{R}^3_+$

Following the stationary scattering theory (e.g. Isozaki [8]), we arrive at the characterization of $\mathcal{B}^*$-solutions to $L_0\mathbf{u} = \lambda\mathbf{u}$.

Theorem 2.6. *Let $\lambda > 0$. Suppose that* $\mathbf{u}$ *satisfies*

$$(L_0 - \lambda)\mathbf{u} = \mathbf{0}.$$

Then $\mathbf{u} \in \mathcal{B}^*$ *if and only if*

$$\mathbf{u} = U_0(\lambda)^* \Theta$$

for some $\Theta \in \boldsymbol{h}$.

From Theorem 2.3, we obtain an asymptotic expansion of the $\mathcal{B}^*$-solution of $(L_0 - \lambda)\mathbf{u} = 0$ in terms of spherical waves. Recall that $\varphi_* = (\varphi_1, \varphi_2)$, where φ_j are the components of the unit vector φ to the half-sphere $\mathbb{S}^2_+$ and that $\varphi_- \in \mathbb{S}^2_+$, and $\varphi_{PS} \in \mathbb{S}^2_+$ are defined as

$$\varphi_- = (-\varphi_*, \varphi_3), \qquad \varphi_{PS} = (c_{PS}\varphi_*, [c_{PS}\varphi_*]),$$

similarly for φ_{SP}.

Corollary 2.1. *Let $\lambda > 0$. Suppose that* $\mathbf{u} \in \mathcal{B}^*$ *satisfies* $L_0\mathbf{u} = \lambda\mathbf{u}$. *Then we have*

$$\begin{aligned}
\boldsymbol{u}(x) \simeq\ & \frac{C_{\rho_0}}{\sqrt{\lambda_P}} \frac{e^{i\sqrt{\lambda_P}r}}{r} \left\{ (\eta_P^{(3)} g_P)(\varphi) + c_{SP}^2 \eta_P^{(2)}(\varphi) g_{SV}(\varphi_{SP}) \right\} \mathbf{d}_P(\varphi) \\
& + \frac{C_{\rho_0}}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r} \left\{ \chi(\mathbb{S}_{SV}) c_{PS}^2 \eta_{SV}^{(3)}(\varphi) f_P(\varphi_{PS}) \right. \\
& \left. + \left(\chi(\mathbb{S}_{SV}) \eta_{SV}^{(2)}(\varphi) + \chi(\mathbb{S}_{SV}^0) \eta_{SV}^{(5)}(\varphi) \right) f_{SV}(\varphi) \right\} \hat{\mathbf{d}}_{SV}(\varphi) \\
& + \frac{C_{\rho_0}}{\sqrt{\lambda_S}} \frac{e^{i\sqrt{\lambda_S}r}}{r} \left(g_{SH} \mathbf{d}_{SH} \right)(\varphi) \\
& + D \sum_{\ell=1}^{2} \frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} e^{-\sqrt{\lambda_R}\tau_\ell x_3} R_\ell \left(g_R \mathbf{d}_R^{(\ell)} \right)(\varphi_*) \\
& + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_P}} \frac{e^{-i\sqrt{\lambda_P}r}}{r} J\left(g_P \hat{\mathbf{d}}_P \right)(\varphi) + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r} J\left(g_{SV} \mathbf{d}_{SV} \right)(\varphi) \\
& + \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_S}} \frac{e^{-i\sqrt{\lambda_S}r}}{r} J\left(g_{SH} \mathbf{d}_{SH} \right)(\varphi) + \overline{D} \sum_{\ell=1}^{2} \frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}} e^{-\sqrt{\lambda_R}\tau_\ell x_3} R_\ell J\left(g_R \mathbf{d}_R^{(\ell)} \right)(\varphi_*)
\end{aligned}$$

for some $(g_P, g_{SV}, g_{SH}, g_R) \in \boldsymbol{h}$, *where*

$$C_{\rho_0} = \frac{e^{-i\pi/2}}{\sqrt{2\pi\rho_0}}, \quad D = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \mathcal{N}_R, \qquad \tau_1 = \tilde{c}_{RP}, \quad \tau_2 = \tilde{c}_{RS}.$$

This result shows that any $\mathcal{B}^*$-solution of $L_0\mathbf{u} = \lambda\mathbf{u}$ is approximated in terms of outgoing spherical body waves and Rayleigh surface wave, and incoming spherical body waves and Rayleigh surface wave; outgoing P-wave is described as a sum of reflected wave generated by the incident P-wave and reflected wave generated by the incident SV-wave, and outgoing SV-wave is also described as a sum of reflection waves. This is consistent with the phenomenon known as the seismic wave propagation.

Our next result provides an asymptotic representation of the $\mathcal{B}^*$-solution uniform in a neighborhood of the critical circle $\partial\mathbb{S}_{SV}$. We see that the smooth transition near the critical circle is described by the Fresnel type integral $\mathrm{Fr}(t)$:

$$\mathrm{Fr}(t) = \int_t^\infty e^{-is^2}\,ds.$$

We set

$$\psi(s,\varphi) = |\varphi_*|s + \varphi_3\sqrt{1-s^2}, \quad \varphi = (\varphi_*, \varphi_3) \in \mathbb{S}^2_+$$
$$\alpha(|\varphi_*|) = \mathrm{sgn}(c_{SP} - |\varphi_*|)\sqrt{1-\psi(c_{SP},\varphi)}, \quad \beta(s) = \sqrt{1-s},$$
$$\mathfrak{Fr}(x,s) = \int_{x\alpha(s)}^{x\beta(s)} e^{-it^2}\,dt$$

for $0 < s < 1$, where $\mathrm{sgn}(x)$ denotes the signum function.

Theorem 2.7. *Let $\lambda > 0$. Suppose that $\mathbf{u} \in \mathcal{B}^*$ satisfies $L_0\mathbf{u} - \lambda\mathbf{u} = \mathbf{0}$. Then for some $(g_P, g_{SV}, g_{SH}, g_R) \in \boldsymbol{h}$, we have*

$$\begin{aligned}
\mathbf{u} \simeq{} & \frac{C_{\rho_0}}{\sqrt{\lambda_P}}\frac{e^{i\sqrt{\lambda_P}r}}{r}\left\{(\eta_P^{(3)}g_P)(\varphi) + c_{SP}^2\eta_P^{(2)}(\varphi)g_{SV}(\varphi_{SP})\right\}\mathbf{d}_P(\varphi) \\
&+ \frac{C_1}{\sqrt{\lambda_S}}\frac{e^{i\sqrt{\lambda_S}r}}{r}\left\{c_{PS}^2\mathrm{Fr}\left(-\sqrt{r}\lambda_P^{1/4}\alpha(|\varphi_*|)\right)\eta_{SV}^{(3)}(\varphi)g_P(\varphi_{PS})\right. \\
&\left.+\left(\mathrm{Fr}\left(-\sqrt{r}\lambda_S^{1/4}\alpha(|\varphi_*|)\right)\eta_{SV}^{(2)}(\varphi) + \mathfrak{Fr}(\sqrt{r}\lambda_S^{1/4},\varphi_*)\eta_{SV}^{(5)}(\varphi)\right)g_{SV}(\varphi)\right\}\hat{\mathbf{d}}_{SV}(\varphi) \\
&+ \frac{C_{\rho_0}}{\sqrt{\lambda_S}}\frac{e^{i\sqrt{\lambda_S}r}}{r}(g_{SH}\mathbf{d}_{SH})(\varphi) + D\sum_{\ell=1}^2\frac{e^{i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}e^{-\sqrt{\lambda_R}\tau_\ell x_3}R_\ell\left(g_R\mathbf{d}_R^{(\ell)}\right)(\varphi_*) \\
&+ \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_P}}\frac{e^{-i\sqrt{\lambda_P}r}}{r}J(g_P\hat{\mathbf{d}}_P)(\varphi) \\
&+ \frac{\overline{C_1}}{\sqrt{\lambda_S}}\frac{e^{-i\sqrt{\lambda_S}r}}{r}\overline{\mathrm{Fr}\left(-\sqrt{r}\lambda_S^{1/4}\alpha(|\varphi_*|)\right)}J(g_{SV}\eta_{SV}^{(1)}\mathbf{d}_{SV})(\varphi) \\
&+ \frac{\overline{C_1}}{\sqrt{\lambda_S}}\frac{e^{-i\sqrt{\lambda_S}r}}{r}\overline{\mathfrak{Fr}(\sqrt{r}\lambda_S^{1/4},\varphi_*)}J(g_{SV}\mathbf{d}_{SV})(\varphi) \\
&+ \frac{\overline{C_{\rho_0}}}{\sqrt{\lambda_S}}\frac{e^{-i\sqrt{\lambda_S}r}}{r}J(g_{SH}\mathbf{d}_{SH})(\varphi) \\
& \overline{D}\sum_{\ell=1}^2\frac{e^{-i\sqrt{\lambda_R}r_*}}{\sqrt{r_*}}e^{-\sqrt{\lambda_R}\tau_\ell x_3}R_\ell\left(g_R\mathbf{d}_R^{(\ell)}\right)(-\varphi_*)
\end{aligned}$$

where $r = |x|$, $\varphi = x/r$, $r_* = |x_*|$, $\hat{\varphi}_* = x_*/r_*$ and

$$C_{\rho_0} = \frac{e^{-i\pi/2}}{\sqrt{2\pi\rho_0}}, \quad C_1 = \frac{e^{i\pi/4}}{\sqrt{\pi}}C_{\rho_0}.$$

Remark 2. Theorem 2.7 implies Corollary 2.1 due to the asymptotics of the Fresnel type integrals:

$$\mathrm{Fr}(-\sqrt{r}\lambda_P^{1/4}\alpha(|\varphi_*|)) = \sqrt{\pi}e^{-i\pi/4}\chi(\mathbb{S}_{SV}) + O\left(\frac{1}{r}\right), \qquad r = |x| \to \infty$$

for any fixed $\varphi = \in \mathbb{S}^2_+ \setminus \partial\mathbb{S}_{SV}$. Noting that if $\varphi \in \partial\mathbb{S}_{SV}$, then $\alpha(|\varphi_*|) = 0$, we see that

$$\mathrm{Fr}(-\sqrt{r}\lambda_P^{1/4}\alpha(|\varphi_*|)) = \mathrm{Fr}(0) = \frac{\sqrt{\pi}}{2}e^{-i\pi/4}$$

for $\varphi \in \partial\mathbb{S}_{SV}$, which describes the asymptotic approximation on the critical circle $\partial\mathbb{S}_{SV}$ for the reflected S-wave generated by incident P-wave. Thus, our expansion describes the complicated wave phenomenon near the critical circle. It can be also observed that the leading term of the asymptotic expansion of $\mathcal{B}^*$-solution in a neighborhood of the boundary $x_3 = 0$ is the Rayleigh surface wave of the form $e^{-\sqrt{\lambda}x_3}e^{-i\sqrt{\lambda}r_*}/\sqrt{r_*}$; the leading term of them away from the boundary $x_3 = 0$ is the body wave of the form $e^{-i\sqrt{\lambda}r}/r$. Hence, Theorem 2.7 gives the uniform asymptotic expansion with respect to directions.

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